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On the Minimally Non-outerplanarity of Generalized Middle and Total Graphs

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Abstract

Systo and Topp introduced the notions of generalized line, middle and total graphs and they studied the planarity and outerplanarity of these classes of graphs. Conditions under which generalized middle graphs and generalized total graphs are minimally non-outerplanar are discussed in this paper.

Keywords: Planar graphs, middle graphs, total graphs, regular graphs, cocktail party graphs, cliques, cutvertices.

1. Introduction

Let $V(G)$, $E(G)$, and $L(G)$denote the vertex set, edge set and line graph of a graph G, respectively. The union and join (or sum) of two graphs G and H are denoted by $G \cup H$ and $G + H$ respectively.[1] The middle graph $M(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices of $M(G)$ are adjacent if they are adjacent edges of G or one is a vertex and the other is an edge of G incident with it.[4] The total graph $T(G)$ of a graph G is the graph whose vertices can be put in one-to-one correspondence with the set of vertices and edges of G in such a

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way that two vertices of $T(G)$ are adjacent if and only if the corresponding elements of G are adjacent (if both elements are vertices or both are edges) or incident (if one element is a vertex and the other is an edge).[1,3] For any non-negative integer n, a cocktail party graph $CP(n)$, is defined as the unique $(2n-2)$-regular graph of order $2n$. For the sake of simplicity, we assume that $CP(0)$ exists and is $(0,0)$ graph, without vertices. Most of the definitions and notations not specifically here are used in the sense of Harary.[1]

Hoffman, Rao *et al*. and Systo *et al*. studied planarity (outerplanarity) of the generalized line, middle and total graphs.[2, 5, 6] We shall restrict ourselves to non-trivial connected graphs throughout this paper. For a graph G, let $L(G, v)$ be the clique on the set of all edges incident with the vertex v in G and N^* denotes the set of non-negative integers. For a function $f: V(G) \to N^*$, let $\{CP(f(v)) : v \in V(G)\}$ be the family of cocktail party graphs disjoint from each other and from G and $L(G)$. The notions of the generalized line, middle and total graphs are as in [6].

1. The generalized line graph $L(G, f)$ of G is defined as $\cup_{v \in V(G)}\{L(G, v) + CP(f(v))\}$.
2. The generalized middle graph $M(G, f)$ of G is defined as $\cup_{v \in V(G)}\{L(G, v) + [CP(f(v)) \cup < v >]\}$.
3. The generalized total graph $T(G, f)$ of G is defined as $\cup_{v \in V(G)}\{L(G, v) + [CP(f(v)) \cup < v >]\} \cup G$,

where$< v >$ denotes a one-vertex graph K_1 on the vertex v_1. Notice that $L(G, f)$ is a subgraph of $M(G, f)$ and $M(G, f)$ is a subgraph of $T(G, f)$. Further, $L(G, f) = L(G)$ [or $M(G, f) = M(G)$ or $T(G, f) = T(G)$] if and only if $f(v) = 0$ for every vertex v of G.

A set of vertices of a planar graph G is called an inner vertex set $I(G)$ of G, if G can be drawn on the plane in such a way that each vertex in $I(G)$ is incident only with the interior faces of G and $I(G)$ contains the minimum possible number of vertices of G. Each vertex in $I(G)$ is said to be an inner vertex of G. The number $|I(G)|$ in such an embedding of G is denoted by $i(G)$ *and* is called the inner vertex number of G. Clearly, G is outerplanar if and only if $i(G) = 0$ and G is minimally non-outerplanar if $i(G) = 1$.[4]

In this paper, we study the minimally non-outerplanarity of generalized middle and total graphs. Our results on generalized middle graphs extend to those for middle graphs given in [4].

2. Generalized Middle Graphs

We begin with the theorem from [6] as a remark.

Remark 2.1

For a nontrivial connected graph G and a function $f: V(G) \to N^*$, the generalized middle graph $M(G, f)$ is planar (outerplanar) if and only if G is planar (outerplanar) with $d_G(v) + f(v) \leq 3$ $(d_G(v) + f(v) \leq 2)$ for every vertex v of G.

The following theorem gives a criterion for the generalized middle graph of a graph to be minimally non-outerplanar.

Theorem 2.2

For a nontrivial connected graph G and a function $f: V(G) \to N^*$, the generalized middle graph $M(G, f)$ is minimally non-outerplanar if and only if the following conditions hold true.

a) G is planar,

b) $d_G(v) + f(v) \leq 3$, for each vertex v of G and

c) $d_G(v) + f(v) = 3$, v is the unique cutvertex in G and furthermore, $d_G(u) + f(u) \leq 2$ for all other vertices $u \neq v$ in G.

Proof

Suppose that $M(G, f)$ is minimally non-outerplanar and hence, it is planar. By Remark 2.1, G is planar and $d_G(v) + f(v) \leq 3$, for each vertex v of G. Also, $M(G, f)$ is not outerplanar. In view of Remark 2.1, $d_G(v) + f(v) \geq 3$. Consequently, G must contain a vertex v such that $d_G(v) + f(v) = 3$.

Now, we prove the uniqueness of the vertex v. For this, assume that G has at least two vertices v_1 and v_2 such that $d_G(v_i) + f(v_i) = 3$ for $i = 1, 2$. There are three cases to discuss:

Case 1

If $d_G(v_i) = 3$ for $i = 1, 2,$ then each vertex v_i and three of its neighbouring vertices lie in a subgraph of G isomorphic to $K_{1,3}$, whose middle graph $M(K_{1,3})$ contains the wheel W_4 as its subgraph. Consequently, $M(G)$ has at least two edge –disjoint subgraphs, each is isomorphic to W_4. Since $i(W_4) = 1$, $i(M(K_{1,3})) \geq 1$, immediately, $M(G, f)$ contains a subgraph isomorphic to $2W_4$, and hence $i(M(G, f)) \geq 2$, a contradiction.

Case 2

Assume $d_G(v_1) = 3$ and $d_G(v_2) = 2$. Since $d_G(v_1) = 3$, arguing as above, we see that $M(G, f)$ has a non-outerplanar subgraph isomorphic to W_4, and hence $i(M(G, f)) \geq 1$. Since $d_G(v_2) = 2$, it follows that $f(v_2) = 1$ because $d_G(v_2) + f(v_2) = 3$ in G. It is easy to see that $M(G, f)$ has another non-outerplanar subgraph isomorphic to $L(G,\ v_2) + \{CP(1) \cup < v_2 >\}$ and in particular, this subgraph contains a proper subgraph isomorphic to $K_{2\,,3}$ (see Figure 1). But $i\big(K_{2\,,3}\big) = 1$. Now, $M(G, f)$ contains a subgraph isomorphic to $K_{2\,,3} \cup w_4$ and it has at least two inner vertices, contradicting the minimally non-outerplanarity of $M(G, f)$.

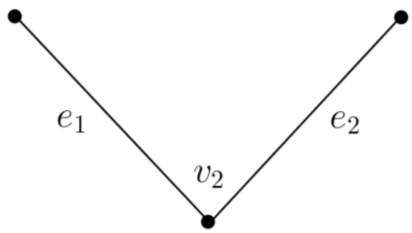
Case 3

Let $d_G(v_i) = 2$ for $i = 1$ and 2. Arguing as above, we can show that $M(G, f)$ contains two distinct non-outerplanar subgraphs, each of which is isomorphic to $K_{2\,,3}$. This implies that $i(M(G, f)) \geq 2$, a contradiction.

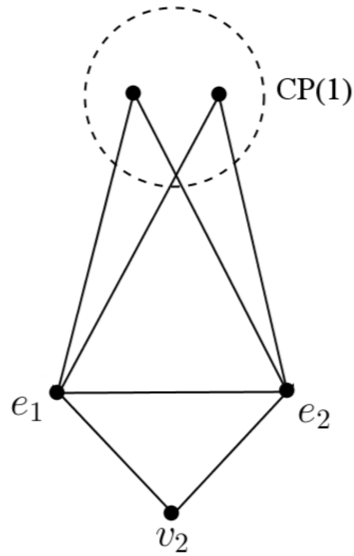
In all cases, we arrived at a contradiction. It follows that G contains a unique vertex v such that $d_G(v) + f(v) = 3$ and all other vertices $u \neq v$ satisfy$d_G(u) + f(u) \leq 2$.

Finally, we show that the vertex v mentioned above, is a cutvertex of G. If $d_G(v) = 3$ and v is a unique vertex of degree 3 (because $f(v) = 0$ in this case), it follows that v is a cutvertex. Next, assume that $d_G(v) = 2$ and v is a non-cutvertex. Then $f(v) = 1$ and v lies on a cycle say $C_n : v, v_1, v_2, \dots . v_{n-1}, v$ for $n \geq 3$. As before, $M(C_n, f)$ contains a non-outerplanar subgraph isomorphic to $L(C_n, v) + \{CP(1) \cup < v >\} \cup C_n$ and it has a proper subgraph homeomorphic to $K_{2,4}$ with $i(K_{2,4}) = 2$ (Figure 2). It follows that $i(M(C_n, f)) \geq 2$. Since $M(C_n, f)$ is a non-outerplanar subgraph of $M(G, f)$, $M(G, f)$ has at least two inner vertices, a contradiction. Thus, v is a cutvertex.

Conversely, assume that G and f satisfy given conditions (a), (b) and (c). Then it is not hard to check that $M(G, f)$ is minimally non-outerplanar. ■



$d_G(v_2) = 2$; $e_i (i = 1,2)$ an edge at v_2



$L(G, v_2) + \{CP(1) \cup < v_2 >\}$

Fig. 1

3. Generalized total graphs

It is shown in [3] that total graphs are not minimally non-outerplanar. In this section, we obtain a criterion for the generalized total graph $T(G,f)$ of a graph G to be minimally non-outerplanar when f is non-zero. Now, we consider the theorem from [6] as the remark.

Remark 3.1

For a non-trivial connected graph G and a function $f: V(G) \rightarrow N^*$, the generalized total graph $T(G,f)$ is planar (outerplanar) if and only if the following conditions hold:

a) $d_G(v) + f(v) \leq 3$ ($d_G(v) + f(v) \leq 2$) for each vertex v of G and

b) $d_G(v) = 3$ ($d_G(v) = 2$), v is a cutvertex of G.

Theorem 3.2

For a non-trivial connected graph G and a non-zero function $f: V(G) \rightarrow N^*$, the generalized total graph $T(G,f)$ is minimally non-outerplanar if and only if the following conditions hold:

a) G is a path of order $n \geq 3$,

b) $d_G(v) + f(v) \leq 3$ for every vertex v of G and

c) $d_G(v) + f(v) = 3$, v is the unique vertex of degree 2 in G and furthermore, $d_G(u) + f(u) \leq 2$ for all other vertices $u \neq v$ in G.

Proof

Suppose $T(G,f)$ is minimally non-outerplanar and hence, it is planar. From Remark 3.1, $d_G(v) + f(v) \leq 3$ for each vertex v of G and if $d_G(v) = 3$, then v is a cutvertex of G. Since $T(G,f)$ is not outerplanar, by application of Remark 3.1, G has a vertex v such that $d_G(v) + f(v) = 3$. Assume that the vertex v has degree 3, then $f(v) = 0$. Moreover, v and three of its neighbouring vertices lie in a subgraph of G isomorphic to $K_{1,3}$, whose total graph $T(K_{1,3})$ contains at least two inner vertices as $i(T(K_{1,3})) \geq 2$).[3] Since $T(K_{1,3})$ is a subgraph of $T(G,f)$, we have $i(T(G,f)) \geq 2$, contradicting the minimally non-outerplanarity of $T(G, f)$. This implies that every vertex of G has degree at most two. Since G is nontrivial and connected, either G is a path of order at least two, or a cycle. If $\Delta(G) = 1$, then $G = K_2$ and $f(v) = 2$ for each vertex v of G. It is easy to see that $T(G, f)$ has a non-outerplanar subgraph isomorphic to $K_1 + (2K_1 \cup C_4)$ and it has two inner vertices, a contradiction.

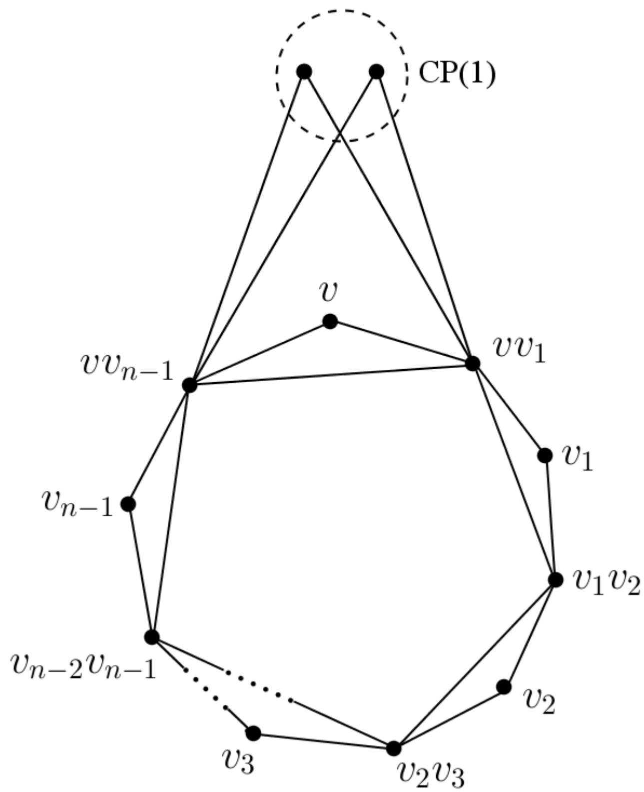


Fig. 2. $M(C_n,\ f)$ with $f(v) = 1$.

If $\Delta(G) = 2$, then $G = P_n$ or $G = C_n$ for $n \geq 3$. Suppose G is C_n : $v, v_1, v_2, \ldots . v_{n-1}, v$ for $n \geq 3$. Then we have $f(v) = 1$. As usual, $T(G, f)$ contains a non-outerplanar subgraph homeomorphic to $K_{2,4}$ (Figure 2). Consequently, $T(G, f)$ has at least two inner vertices. This is a contradiction. So, $G = P_n$ for $n \geq 3$ is the possible case.

Finally, we shall prove the uniqueness of the vertex. For this, assume that G has at least two vertices $v_1$and v_2, each has degree 2 in G. We have $f(v_1) = f(v_2) = 1$. By arguing in the similar manner as in Case 3 of Theorem 2.2, we arrived at the conclusion that $i(T(G, f)) \geq 2$. This is a contradiction. Thus, G has a unique vertex of degree 2 such that $d_G(v) + f(v) = 3$. In addition, if $d_G(u) + f(u) \geq 3$ for some vertex $u \neq v$ in G, then it is easy to see that $i\big(T(G, f)\big) > 1$, again a contradiction.

Conversely, suppose that conditions (a), (b) and (c) hold for G. It is easy to verify that $T(G, f)$ is minimally non-outerplanar. ■

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